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On Binary Sextics with Linear Transformations into Themselves.

By Oskar Bolza.

In the following investigation, which I have undertaken at the suggestion of Prof. F. Klein, I consider those binary sextics which remain unchanged (or are only changed by a constant factor) for certain linear transformations of the variables. In the first section I determine all the binary sextics of this special character; in the second section the relations are established which in these cases exist between the rational absolute invariants of the sextic; and a completion of a theorem of Clebsch's is given, concerning the possibility of transformation of one sextic into another by linear substitution. Finally, in the third section I examine the corresponding relations between the transcendental absolute invariants, that is to say, between the ϑ-moduli $\tau_{11}, \tau_{12}, \tau_{22}$.

The subject is of interest for the theory of the hyperelliptic modular functions; for if we represent a system of values of the three complex magnitudes $\tau_{11}, \tau_{12}, \tau_{22}$ by means of a point in a space of six dimensions, it can be shown that the aforesaid relations between the τ_{ik} represent the edges and vertices of the fundamental region ("Fundamentalraum") for the hyperelliptic modular functions. And the question, under what conditions a sextic can be transformed into another by linear substitution, is closely connected with the question whether the rational absolute invariants are single-valued ("eindeutig") functions of the variables $\tau_{11}, \tau_{12}, \tau_{22}$.

Section I.

Determination of all the Binary Sextics with Linear Transformations into Themselves.

§1.

I say, for shortness, that a sextic $f(x_1, x_2)$ remains unchanged for the substitution

$$\left.\begin{aligned} x_1 &= px_1' + qx_2' \\ x_2 &= p'x_1' + q'x_2' \end{aligned}\right\} \qquad \text{(S)}$$

if
$$f(x_1, x_2) = f'(x_1', x_2') = cf(x_1', x_2'), \tag{1}$$
c being any constant factor.

It is clear that the entire system of the substitutions which leave a given sextic f unchanged, form *a group*, G; we shall show that this group is *finite*, provided the substitutions S be reduced to the determinant 1; that is,

$$pq' - p'q = 1. \tag{2}$$

Proof. The roots α_i of the equation $f(x_1, x_2) = 0$, which we suppose to be all different from each other, are connected with the roots α_i' of the equation $f'(z_1', z_2') = 0$ by the relations

$$\alpha_i' = \frac{q'\alpha_i - q}{-p'\alpha_i + p}\,;\ (i = 1,\ 2, \ldots . 6); \tag{3}$$

accordingly each root α_i' corresponds to one of the roots α_i. Now, if the substitution S leaves the sextic f unchanged, the roots α_i' must be identical with the α_i, only in different order; therefore, to each substitution S which leaves f unchanged, corresponds a definite permutation,

$$\sigma = \begin{pmatrix} \alpha_i \\ \alpha_i' \end{pmatrix}$$

of the roots α_i, which is determined by the equations (3).

It is easy to see that the permutations σ corresponding to the linear substitutions S likewise form a group, Γ, which is isomorphic with the group G, and to each substitution of G corresponds a single permutation of Γ.

Conversely, to the permutation 1 of the group Γ correspond all the substitutions of G which satisfy the conditions

$$\alpha_i' = \alpha_i,$$

that is,
$$\alpha_i = \frac{q'\alpha_i - q}{-p'\alpha_i + p},$$

or,
$$-p'\alpha_i^2 + (p - q')\,\alpha_i + q = 0,$$

for $i = 1,\ 2 \ldots . 6$.

This equation, considered as an equation with the unknown α_i, can be satisfied in two different ways:

1). The coefficients may not be all $= 0$. In this case the equation $f = 0$ has only two distinct roots, and f is reducible by linear substitution to the form

$$f = z_1^{\lambda_1} z_2^{\lambda_2},\ (\lambda_1 + \lambda_2 = 6);$$

here the group of substitutions which leave f unchanged is evidently infinite.

2). All the coefficients may be $= 0$; that is,

$$p' = 0, \; p = q', \; q = 0;$$

therefore, because of (2),

$$p = +1, \; q' = +1, \text{ or } p = -1, \; q' = -1.$$

That is to say, to the permutation 1 in the group Γ correspond in the group G the two substitutions

$$x_1 = +x_1', \; x_2 = +x_2', \text{ and } x_1 = -x_1', \; x_2 = -x_2'.$$

Excepting the trivial case where $f = z_1^{\lambda_1} z_2^{\lambda_2}$, we have therefore the result:

The group Γ, which contains certainly only a finite number of distinct operations, is hemiedrically isomorphic with the group G of homogeneous substitutions of the determinant 1, and it is holoedrically isomorphic with the corresponding group of non-homogeneous[*] substitutions. Thence follows immediately that the group G is a finite group, which was to be proved.

§2.

The group G being a finite group of binary substitutions, must be one of the following well-known groups:[†]

1). The self-evident group $x_1 = \pm x_1'$, $x_2 = \pm x_2'$.
2). A cyclic group.
3). A diedron group.
4). The tetraedron group.
5). The oktaedron group.
6). The ikosaedron group.

The group $x_1 = \pm x_1'$, $x_2 = \pm x_2'$ leaves *every* binary sextic unchanged; to each of the other groups belong three canonical invariant groundforms; denoting them in the notation of Klein by F_1, F_2, F_3, any binary quantic which remains unchanged for the group in question is reducible by linear transformation to the form[‡]

$$F_1^{\alpha} F_2^{\beta} F_3^{\gamma} \Pi_i \left(A^{(i)} F_1^{\nu_1} + B^{(i)} F_2^{\nu_2}\right), \tag{4}$$

the $A^{(i)}$, $B^{(i)}$ being any constant factors; the α, β, γ, ν_1, ν_2 being integers.

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[*] Concerning the difference between homogeneous and non-homogeneous substitutions cf. F. Klein, Vorlesungen über das Ikosaeder, p. 44.

[†] Cf. F. Klein, l. c. p. 115.

[‡] Cf. F. Klein, l. c. p. 49.

We have, then, only to seek, for each of the above enumerated groups, all the cases where the expression (4) represents a sextic.

Excluding the cases where f has equal roots, we easily find the following result:

Any binary sextic f with linear transformations into itself (except of course the transformations $x_1 = \pm x_1'$, $x_2 = \pm x_2'$), and whose roots are all different from each other, is reducible, by a linear substitution, *to one of the following canonical forms:*

$$\left.\begin{array}{rl}
\text{I.} & f = z_1^6 + \alpha z_1^4 z_2^2 + \beta z_1^2 z_2^4 + z_2^6 \\
 & \qquad\qquad (\textit{Cyclic group } n = 2) \\
\text{II.} & f = z_1 (z_1^5 + z_2^5) \\
 & \qquad\qquad (\textit{Cyclic group } n = 5) \\
\text{III.} & f = z_1 z_2 (z_1^4 + \alpha z_1^2 z_2^2 + z_2^4) \\
 & \qquad\qquad (\textit{Diedron group } n = 2) \\
\text{IV.} & f = z_1^6 + \alpha z_1^3 z_2^3 + z_2^6 \\
 & \qquad\qquad (\textit{Diedron group } n = 3) \\
\text{V.} & f = z_1^6 + z_2^6 \\
 & \qquad\qquad (\textit{Diedron group } n = 6) \\
\text{VI.} & f = z_1 z_2 (z_1^4 + z_2^4) \\
 & \qquad\qquad (\textit{Oktaedron group})
\end{array}\right\} \tag{A}$$

The inequalities for the parameters α, β which must be added in order to discriminate between the different cases, are given in the following section.

SECTION II.

The Relations between the Rational Invariants.

§3.

We now purpose to find the necessary and sufficient criteria that a given binary sextic be reducible by linear transformation to one of the above enumerated canonical forms.

These special sextics have been examined by Clebsch in his "Theorie der binaeren algebraischen Formen," and more recently by Maisano in a paper "Sulla sestica binaria" (Atti della R. Accademia dei Lincei, 1883–84).

For the first two cases the criteria in question are completely given by Clebsch in the form of relations between the invariants of the sextic. Using Clebsch's notation,* they are

$$\begin{aligned} &\text{I. } f = z_1^6 + \alpha z_1^4 z_2^2 + \beta z_1^2 z_2^4 + z_2^6, \\ &\quad R = 0, \ A_{ll} A_{mm} - A_{lm} \gtrless 0.\dagger \qquad (5) \\ &\text{II. } f = z_1 (z_1^5 + z_2^5), \\ &\quad A = 0, \ B = 0, \ C = 0, \ D \gtrless 0.\ddagger \qquad (6) \end{aligned}$$

The criteria in the other cases are given by Maisano as the vanishing of certain *covariants*. But in these cases also they can be expressed by relations between *invariants*, as we shall show.

The subject is closely connected with the determination of the conditions that a sextic f be transformable into another f' by linear substitution. In general, the equality of the corresponding rational absolute invariants of f and f' is necessary *and sufficient*, as Clebsch has proved§ with the aid of the typical representation ("typische Darstellung") of the binary sextic by means of quadratic covariants. But whether the same condition is sufficient in those exceptional cases where the typical representation is impossible, has not yet been investigated, so far as I know. And as the canonical forms III to VI of our table (A) are the most important of these exceptional cases, I shall answer this question in what follows. We denote with Clebsch:

$$\left.\begin{aligned} &f = a_x^6 = a_0 x_1^6 + 6a_1 x_1^5 x_2 + 15 a_2 x_1^4 x_2^2 + 20 a_3 x_1^3 x_2^3 + 15 a_4 x_1^2 x_2^4 + 6 a_5 x_1 x_2^5 + a_6 x_2^6, \\ &i = (ab)^4 a_x^2 b_x^2 = \alpha_0 x_1^4 + 4\alpha_1 x_1^3 x_2 + 6\alpha_2 x_1^2 x_2^2 + 4\alpha_3 x_1 x_2^3 + \alpha_4 x_2^4, \\ &l = (ai)^4 a_x^2, \ m = (il)^2 i_x^2, \ n = (im)^2 i_x^2, \\ &A = (ab)^6, \ B = (ii')^4, \ C = (ii')^2 (i'i'')^2 (i''i)^2, \\ &D = (nl)^2, \ R = -(lm)(mn)(nl). \end{aligned}\right\} \quad (7)$$

Moreover,

$$\left.\begin{aligned} A_{ll} &= (ll')^2 = 2C + \frac{1}{3} AB, \\ A_{lm} &= (lm)^2 = \frac{2}{3} (B^2 + AC), \\ A_{ln} &= (ln)^2 = D = (mm') = A_{mm}, \\ A_{mn} &= (mn)^2 = \frac{1}{2} B A_{lm} + \frac{1}{3} C A_{ll}, \\ A_{nn} &= (nn')^2 = \frac{1}{2} BD + \frac{1}{3} C A_{lm}. \end{aligned}\right\} \quad (8)$$

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* Cf. the following page.

† Cf. Clebsch, l. c. p. 455.

‡ Cf. Clebsch, l. c. p. 457.

§ L. c. p. 421.

The typical representation is then, and only then, impossible when the three quadratic covariants l, m, n have a common factor. According to the investigations of Clebsch and Maisano, this can happen in two different ways; either l vanishes identically or, l being different from zero, m differs from l only by a constant factor.*

§4.

Let us first examine the case

$$l \equiv 0. \tag{9}$$

We then have from the formulæ (8),

$$A_{ll} = 0,\ A_{lm} = 0,\ D = 0. \tag{10}$$

Conversely, from these three equations follows $l \equiv 0$. For A_{ll} being $= 0$, we may put

$$l = l_0 x_1^2.$$

If l_0 were different from zero, we should infer from

$$A_{lm} = l_0 m_2 = 0,$$

that $m_2 = 0$,

and from $D = -2m_1^2 = 0$, that $m_1 = 0$.

Therefore we should have

$$m = l_0 \alpha_2 x_1^2;$$

that is to say, m would differ from l only by a constant factor. Now, Clebsch has proved the two lemmas,† "If m vanishes identically, l must vanish also," and "If $m =$ const. l, and $l \gtrless 0$, l cannot be a complete square." Thence follows that l_0 must be $= 0$; we have therefore the lemma:

The covariant condition $l \equiv 0$ is equivalent to the invariant conditions

$$A_{ll} = 0,\ A_{lm} = 0,\ D = 0.$$

Now, if for a binary sextic f the covariant l vanishes identically, the following cases are to be distinguished:‡

1). If at the same time $A = 0$,

the sextic f has a quadruple, a quintuple, or a sextuple factor, and the three other invariants vanish also,

$$B = 0,\ C = 0,\ D = 0.$$

2). On the contrary, if $A \gtrless 0$,

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[*] Clebsch, l. c. p. 437, and Maisano, l. c. p. 32.

[†] L. c. p. 445 and p. 441.

[‡] Cf. Maisano, l. c. p. 32.

the sextic f is, by a linear substitution of the determinant 1, reducible to one of the two forms

$$a\,(z_1^6 + z_2^6),\quad a \gtrless 0,$$

or

$$6az_1z_2\,(z_1^4 + z_2^4),\quad a \gtrless 0.$$

In the former case we have besides $B \gtrless 0$, in the latter, $B = 0$.

Herewith are the criteria for the two cases V and VI of the table (A) reduced to relations between invariants, but they admit of further simplification:

a) $$f = a\,(z_1^6 + z_2^6),\quad a \gtrless 0;$$

then we may replace the equation $A_{lm} = 0$ of the system (10) by $6B - A^2 = 0$, and the inequality $B \gtrless 0$ follows from $A \gtrless 0$, so that the criteria for this case are

$$6B - A^2 = 0,\quad 2C + \frac{1}{3}AB = 0,\quad D = 0, \qquad (11)$$
$$A \gtrless 0.$$

b) $$f = 6az_1z_2\,(z_1^4 + z_2^4),\quad a \gtrless 0;$$

here the criteria are $$B = 0,\quad C = 0,\quad D = 0, \qquad (12)$$
$$A \gtrless 0.$$

§5.

We pass to the case where $l \gtrless 0$ and m differs from l only by a constant factor; that is, $$m = kl,\ \text{or},\ (lm)\,l_x m_x \equiv 0; \qquad (13)$$
then, according to the two lemmas of Clebsch's cited in the foregoing paragraph, k is certainly $\gtrless 0$ and the two factors of l are different from each other, therefore $A_{ll} \gtrless 0$; besides, $n = k^2 l$; thence follows

$$DA_{ll} = A_{lm}^2,\quad A_{mn}A_{ll} = A_{lm}D.$$

Conversely, from these two equations and the inequality $A_{ll} \gtrless 0$ follows the relation (13).

For, A_{ll} being $\gtrless 0$, we may write

$$l = 2l_1x_1x_2,\ \text{and}\ l_1 \gtrless 0.$$

Thence follow

$$m = -\,2l_1\,(\alpha_1x_1^2 + 2\alpha_2x_1x_2 + \alpha_3x_2^2),$$
$$n = -\,2l_1\,\big((\alpha_0\alpha_3 - \alpha_1\alpha_2)\,x_1^2 + 4\,(\alpha_1\alpha_3 - \alpha_2^2)\,x_1x_2 + (\alpha_1\alpha_4 - \alpha_2\alpha_3)\,x_2^2\big),$$
$$A_{ll} = -\,2l_1^2,\quad A_{lm} = 4l_1^2\alpha_2,\quad A_{ln} = 8l_1^2\,(\alpha_1\alpha_3 - \alpha_2^2).$$

The equation $DA_{ll} = A_{lm}^2$ then becomes

$$\alpha_1\alpha_3 = 0.$$

We may assume $\alpha_1 = 0$, then we have

$$A_{mn} = 4l_1^2(4\alpha_2^3 + \alpha_0\alpha_3^2), \quad A_{ln} = -8l_1^2\alpha_2^2,$$

therefore the equation $\qquad A_{mn}A_{ll} = A_{lm}D$

becomes $\qquad \alpha_0\alpha_3 = 0,$

and the three covariants l, m, n have the common factor x_2, l being at the same time $\gtrless 0$. Now, Maisano has proved,[*] "If l, m, n have a common factor, they have also the other factor common"; we have then the lemma:

The covariant conditions

$$(lm)\, l_x m_x \equiv 0, \quad l \gtrless 0 \tag{13}$$

are equivalent to the invariant conditions

$$DA_{ll} = A_{lm}^2, \quad A_{mn}A_{ll} = A_{lm}D, \tag{14}$$
$$A_{ll} \gtrless 0.$$

Now, from the relations (13) follows that the sextic f is reducible by a linear substitution of the determinant 1 to one of the three forms[†]

$$20a_3z_1^3z_2^3 + a_5z_1z_2^5 + a_6z_2^6, \quad a_3 \gtrless 0,$$
$$a(z_1^6 + \alpha z_1^3z_2^3 + z_2^6), \quad a\alpha(\alpha^2 + 50) \gtrless 0,\text{‡}$$
$$6az_1z_2(z_1^4 + \alpha z_1^2z_2^2 + z_2^4), \quad a\alpha(9\alpha^2 - 100) \gtrless 0.\text{§}$$

In the first case we easily find $\quad C^2 - \frac{1}{6}B^3 = 0, \quad 2AB - 15C = 0;$

in the second case, $\quad C^2 - \frac{1}{6}B^3 = 0, \quad 2AB - 15C \gtrless 0;$

in the third case, $\quad C^2 - \frac{1}{6}B^3 \gtrless 0.$ $\qquad$ (15)

Therefore the relations (14), together with $C^2 - \frac{1}{6}B^3 = 0$, $2AB - 15C = 0$, are the necessary and sufficient conditions that a sextic be reducible by linear substitution to the form

$$20a_3z_1^3z_2^3 + a_5z_1z_2^5 + a_6z_2^6,$$

and similarly for the two other cases. But these conditions can be further simplified. An easy calculation leads to the following results:

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[*] L. c. p. 34.

[†] Cf. Maisano, l. c. p. 34.

[‡] If $\alpha^2 + 50$ were $= 0$, the sextic would be reducible to $6a'z'_1z'_2(z_1'^4 + z_2'^4)$.

[§] If $9\alpha^2 - 100$ were $= 0$, the sextic would be reducible to $a'(z_1'^6 + z_2'^6)$.

a). For
$$f = 20a_3z_1^3z_2^3 + a_5z_1z_2^5 + a_6z_2^6, \qquad a_3 \gtrless 0, \tag{16}$$

the necessary and sufficient conditions are*
$$3.5^2.B = 2^3.A^2,\ \ 3^2.5^3.C = 2^4.A^3,\ \ 3^2.5^5D = 2^7.A^5, \qquad A \gtrless 0. \tag{17}$$

Moreover,
$$a_3 = \sqrt{-\frac{A}{2^2.5}},$$
a_5 and a_6 are completely independent of the invariants.

b). For
$$f = a\,(z_1^6 + \alpha z_1^3z_2^3 + z_2^6), \qquad a\alpha\,(\alpha^2 + 50) \gtrless 0, \tag{18}$$

the necessary and sufficient conditions are
$$C^2 - \frac{1}{6}B^3 = 0,\ \ 9D - 2B(6C + AB) = 0, \qquad D \gtrless 0,\ \ 2AB - 15C \gtrless 0. \tag{19}$$

Moreover,
$$\left.\begin{aligned} a &= \frac{1}{3}\sqrt{\frac{2AB - 15C}{B}} \\ \alpha &= 10\sqrt{\frac{6C + AB}{15C - 2AB}} \end{aligned}\right\} \tag{20}$$
as from the conditions (19) follow:
$$A_{ll} \gtrless 0,\ \ B \gtrless 0,$$
the expressions of *a* and *α* can never become infinite or indeterminate.

Finally, we have for the covariant *l* the expression
$$l = \sqrt{-2A_{ll}}z_1z_2. \tag{21}$$

c). For
$$f = 6az_1z_2\,(z_1^4 + \alpha z_1^2z_2^2 + z_2^4), \qquad a\alpha\,(9\alpha^2 - 100) \gtrless 0, \tag{22}$$

the necessary and sufficient conditions are
$$\left.\begin{aligned} 3AB^2 - 6BC + 4A^2C - 18D &= 0 \\ 4B^3 + 5ABC + 6C^2 - 3AD &= 0 \\ D \gtrless 0,\ \ C^2 - \frac{1}{6}B^3 &\gtrless 0 \end{aligned}\right\} \tag{23}$$

In this case, the proof being more complicated, we shall indicate it in a few words. The two equations (23) can be written in the form
$$\begin{aligned} 6D &= 2AA_{lm} - BA_{ll}, \\ AD &= 2BA_{lm} + CA_{ll}. \end{aligned}$$

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* These conditions are given by Maisano, l. c. p. 35.

Thence we deduce

$$DA_{ll}=A^2_{lm},\quad -D(6B-A^2)=\frac{3}{2}A_{ll}A_{lm},\quad A_{mn}A_{lm}=D^2.$$

D being supposed $\gtrless 0$, we have also $A_{mn}\gtrless 0$, $A_{lm}\gtrless 0$, and from $DA_{ll}=A^2_{lm}$ also $A_{ll}\gtrless 0$. We may then divide the two equations

$$\begin{aligned}A_{mn}A_{lm}&=D^2,\\ A^2_{lm}&=DA_{ll}\end{aligned}$$

by each other; we thus obtain

$$A_{mn}A_{ll}=DA_{lm}.$$

Therefore the three conditions (14) can be deduced from the two equations (23) and the inequality $D\gtrless 0$. But the conditions (14), together with the inequality $C^2-\frac{1}{6}B^3\gtrless 0$, are sufficient, as we have seen above.

As to the parameters a and α, we have

$$\left.\begin{aligned}a&=\frac{1}{2}\sqrt{\frac{C^2-\frac{1}{6}B^3}{6D}}\\ \alpha&=10\sqrt{\frac{B^2+AC}{2A^2B-3AC-15B^2}}\end{aligned}\right\}\tag{24}$$

For the covariant l we find

$$l=\sqrt{-2A_{ll}z_1z_2}.\tag{25}$$

§6.

We can now answer the question proposed in §3. Suppose f or f' to be one of the above enumerated exceptional sextics for which the typical representation is impossible; and suppose it be possible to find a quantity r different from zero, such that between the invariants of the two sextics the relations exist:

$$A'=r^6A,\quad B'=r^{12}B,\quad C'=r^{18}C,\quad D'=r^{30}D,\tag{26}$$

we must then examine whether these conditions are sufficient for the possibility of transformation of f into f' by linear substitution.

We choose as an example the case where between the invariants of f the relations (19) exist:

$$\begin{gathered}C^2-\frac{1}{6}B^3=0,\quad 9D-2B(6C+AB)=0,\\ D\gtrless 0,\quad 2AB-15C\gtrless 0.\end{gathered}$$

We can then reduce f to the form

$$z_1^6+\alpha z_1^3z_2^3+z_2^6,$$

where

$$\alpha=10\sqrt{\frac{6C+AB}{15C-2AB}}.$$

From (26) we have also between the invariants of f' the relations

$$C'^2-\frac{1}{6}B'^3=0,\ gD'-2B'(6C'+A'B')=0,$$
$$D'\gtrless 0,\ 2A'B'-15C'\gtrless 0,$$

and we can therefore reduce f' to the form

$$z_1'^6+\alpha' z_1'^3z_2'^3+z_2'^6,$$

where

$$\alpha'=10\sqrt{\frac{6C'+A'B'}{15C'-2A'B'}}.$$

But from (26) we have $\alpha'=\pm\alpha.$

We may assume $\alpha'=+\alpha$; for, if α' were $=-\alpha$, we should return to the former case by applying to the variables $z_1'z_2'$ the substitution $z_1'=z_1''$, $z_2'=-z_2''$. Thence we immediately infer that f can be transformed into f' by linear substitution.

Exactly the same considerations are applicable to the cases where f is reducible to one of the canonical forms

$$z_1z_2(z_1^4+\alpha z_1^2z_2^2+z_2^4),$$
$$z_1^6+z_2^6,$$
$$z_1z_2(z_1^4+z_2^4).$$

But the proof evidently fails in the cases where f has a quadruple, a quintuple, or a sextuple linear factor; for it is impossible to discriminate between these three cases by invariant conditions, as in each case we have

$$A=0,\ B=0,\ C=0,\ D=0.$$

Finally, in the case of a triple factor,

$$f=20a_3z_1^3z_2^3+6a_1z_1z_2^5+a_6z_2^6,$$

the equality of the absolute invariants is likewise insufficient. For, as the absolute invariants are, in this case, completely independent of a_5 and a_6, the three cases of—

1). One triple and three simple factors,
2). One triple, one double, and one simple factor,
3). Two triple factors,

give the same values of the absolute invariants of f.

The exceptional cases being thus completely discussed, we may express the theorem concerning the transformation of two sextics into each other in the following form:

Two binary sextics which have no triple, quadruple, quintuple, or sextuple factors, can be transformed into each other by linear substitution, if their corresponding rational absolute invariants are equal; but the equality of the absolute invariants is not sufficient if one of the two sextics has such a multiple factor.

§7.

We now revert to the binary sextics with linear transformations into themselves; the criteria that a sextic be reducible to one of the canonical forms (A) being established in §§3, 4, 5, there still remains to determine *the linear substitutions by means of which this reduction is to be performed.*

In each of the *four first cases* of the table (A) there exists a quadratic covariant which, by introducing the canonical variables z_1, z_2, takes the form

$$\text{Const. } z_1 z_2 .$$

We then obtain the substitution in question by putting z_1 and z_2 equal to the two linear factors of that quadratic covariant, multiplied by suitable constant factors. For the first case the quadratic covariant is $(lm)\, l_x m_x$;* for the second, m;† for the third and fourth, l.‡

In the two last cases all the quadratic covariants vanish identically; a different method must therefore be applied, which is founded on the fact that the determination of the canonical variables z_1, z_2 is identical with the solution of the form-problem ("Formenproblem"§) belonging to the group of linear substitutions in question.

In the case of the *diedron group* $n = 6$ (case V of our table), the absolutely invariant groundforms are ||

$$\Phi_1 = (z_1 z_2)^2, \quad \Phi_2 = \left(\frac{z_1^6 + z_2^6}{2}\right)^2, \quad \Phi_3 = \frac{z_1 z_2 (z_1^{12} - z_2^{12})}{4}. \tag{27}$$

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* Cf. Clebsch, l. c. p. 457 and p. 199.

† *Ibid.* p. 467.

‡ Cf. the equations (21) and (25).

§ By the "Formenproblem" belonging to a finite group G of homogeneous linear substitutions of the variables z_1, z_2, Klein understands the following problem: The values of all the invariant groundforms of the group G being given, to determine the corresponding values of the variables z_1, z_2.

|| Cf. Klein, Vorlesungen über das Ikosaeder, p. 63.

They are found to be covariants of the sextic f, viz.

$$\Phi_1 = \frac{i}{A}, \quad \Phi_2 = \frac{f^2}{2A}, \quad \Phi_3 = \frac{fT}{\sqrt{2Ai}}, \tag{28}$$

where
$$T = (aH)\, a_x^5 H_x^7.$$

To find the canonical variables z_1, z_2, we have therefore to solve the problem: The values of the invariant groundforms of the diedron group $n = 6$ being given, to determine the corresponding values of the variables z_1, z_2, and this is exactly the form-problem belonging to the group in question.

The solution follows immediately from the equations

$$\frac{z_1^6 + z_2^6}{2} = \frac{f}{\sqrt{2A}}.$$

$$\frac{z_1^6 - z_2^6}{2} = \sqrt{\frac{A^2 f^2 - 2i^3}{2A^3}}.$$

For the case of the *oktaedron group* (case VI of our table), the invariant groundforms are*

$$\left.\begin{aligned} \Phi_1 &= [z_1 z_2 (z_1^4 + z_2^4)]^2, \\ \Phi_2 &= z_1^8 - 14 z_1^4 z_2^4 + z_2^8, \\ \Phi_3 &= [z_1^{12} + 33 z_1^8 z_2^4 - 33 z_1^4 z_2^8 - z_2^{12}][z_1 z_2 (z_1^4 + z_2^4)]; \end{aligned}\right\} \tag{29}$$

they are covariants of the sextic f, viz.

$$\Phi_1 = -\frac{f^2}{3A}, \quad \Phi_2 = \frac{6H}{A}, \quad \Phi_3 = \frac{12fT}{A^2}. \tag{30}$$

The determination of z_1 and z_2 can be performed with the aid of the resolvent of the third degree,

$$16A\phi^3 + 6H\phi - \frac{f^2}{3} = 0,$$

the three roots of which are

$$\phi_1 = (z_1 z_2)^2, \quad \phi_2 = \frac{(z_1^2 - i z_2^2)^2}{4i}, \quad \phi_3 = -\frac{(z_1^2 + i z_2^2)^2}{4i},$$

whence z_1 and z_2 can immediately be calculated.

The same method is applicable to the determination of the canonical variables in the other cases of the table (A), but the formulæ are rather complicated.

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* Cf. Klein, l. c. p. 63.

SECTION III.

The Relations between the ϑ-Moduli.

§8.

The conditions between the *algebraic* invariants of our special sextics being established in the foregoing section, we now proceed to determine the corresponding conditions between the *transcendental absolute invariants;* that is to say, between the ϑ-moduli $\tau_{\alpha\beta}$.

Using homogeneous variables, we put, with Klein,[*]

$$\left.\begin{aligned} u_1 &= \int^{z_1} \frac{(z_1 dz_2 - z_2 dz_1)}{\sqrt{f(z_1 z_2)}} \\ u_2 &= \int^{z_2} \frac{(z_1 dz_2 - z_2 dz_1)}{\sqrt{f(z_1 z_2)}} \end{aligned}\right\} \tag{31}$$

the lower limit of the integrals being any fixed point in the Riemann's surface belonging to the irrationality $\dfrac{\sqrt{f(z_1, z_2)}}{z_2^3}$. Let

$$\begin{array}{c|cccc|} u_1 & \omega_{11}, & \omega_{12}, & \omega_{13}, & \omega_{14} \\ u_2 & \omega_{21}, & \omega_{22}, & \omega_{23}, & \omega_{24} \end{array} \tag{32}$$

be a system of simultaneous normal periods[†] for the two integrals, and let K_i be the path of integration for the periods ω_{1i}, ω_{2i}. Putting, then, for shortness

$$p_{ik} = \omega_{1i}\omega_{2k} - a_{1k}\omega_{2i},$$

the ϑ-moduli $\tau_{\alpha\beta}$ of Weierstrass are defined by the equations

$$\tau_{11} = \frac{p_{32}}{p_{12}}, \quad \tau_{12} = \frac{p_{42}}{p_{12}} = \frac{p_{13}}{p_{12}} = \tau_{21}, \quad \tau_{22} = \frac{p_{14}}{p_{12}}. \tag{33}$$

In order to determine the relations which must exist between these magnitudes $\tau_{\alpha\beta}$, if the sextic $f(z_1 z_2)$ be one of our canonical forms (A), we first apply to the variables any linear substitution

$$\left.\begin{aligned} z_1 &= p z_1' + q z_2' \\ z_2 &= p' z_1' + q' z_2' \\ \sqrt{f(z_1, z_2)} &= \sqrt{f'(z_1', z_2')} \end{aligned}\right\} \tag{34}$$

Denoting, then, by u_1', u_2', the integrals

$$\left.\begin{aligned} u_1' &= \int^{z_1'} \frac{(z_1' dz_2' - z_2' dz_1')}{\sqrt{f'(z_1', z_2')}} \\ u_2' &= \int^{z_2'} \frac{(z_1' dz_2' - z_2' dz_1')}{\sqrt{f'(z_1', z_2')}} \end{aligned}\right\} \tag{35}$$

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[*] F. Klein, Hyperelliptische Sigmafunctionen, Math. Annalen, Bd. 27.

[†] "Normale Periodicitätsmoduln," according to Clebsch and Gordan, or "Périodes normales," according to Briot.

and by ω'_{1i} and ω'_{2i} those periods of u'_1 and u'_2 which belong to the path of integration K'_i, corresponding to the path K_i, we have for the magnitudes $\tau'_{\alpha\beta}$ the fundamental equations*

$$\tau'_{\alpha\beta} = \tau_{\alpha\beta}; \tag{36}$$

that is to say, the $\tau_{\alpha\beta}$ are transcendental absolute invariants of the binary sextic f.

The equation (36) holds good whatever the sextic f and the substitution (34) may be. Let us now suppose f to be one of the six canonical forms (A) and (34) one of the substitutions leaving the function f unchanged, so that

$$f'(z'_1, z'_2) = cf(z'_1, z'_2). \tag{37}$$

Assuming for simplicity $c = 1$, the magnitudes

$$\omega'_{11}, \omega'_{12}, \omega'_{13}, \omega'_{14}$$

are a system of normal periods *of the same integral* u_1, differing from

$$\omega_{11}, \omega_{12}, \omega_{13}, \omega_{14}$$

only by the path of integration. Therefore the periods ω'_{1i} can be expressed by the ω_{1i} by means of a linear substitution,

$$\omega'_{1i} = m_{i1}\omega_{11} + m_{i2}\omega_{12} + m_{i3}\omega_{13} + m_{i4}\omega_{14}, \tag{s}$$

and likewise

$$\omega'_{2i} = m_{i1}\omega_{21} + m_{i2}\omega_{22} + m_{i3}\omega_{23} + m_{i4}\omega_{24},$$

the integers m_{ik} satisfying the known bilinear relations.

Thence follows a system of formulæ expressing $\tau'_{\alpha\beta}$ by τ_{11}, τ_{12}, τ_{22}, which, for shortness, we write

$$\tau'_{\alpha\beta} = \phi_{\alpha\beta}(\tau_{11}, \tau_{12}, \tau_{22}). \tag{38}$$

The $\tau_{\alpha\beta}$ being homogeneous functions of order zero of the $\omega_{\alpha\beta}$, we obtain the same equations (38) in the case where the constant factor c in (37) is different from unity. Combining the two equations (36) and (38) we have

$$\tau_{\alpha\beta} = \phi_{\alpha\beta}(\tau_{11}, \tau_{12}, \tau_{22}); \tag{39}$$

that is, three relations between the moduli $\tau_{\alpha\beta}$, and these are exactly the relations we had proposed to find in the beginning of this paragraph. Interpreting the relations (39) in a little different way, we have the following proposition:

If the sextic f remains unchanged for a group of linear substitutions of the variables z_1, z_2, any system of ϑ-moduli $\tau_{11}, \tau_{12}, \tau_{22}$ belonging to $\sqrt{f}$ remains likewise unchanged for a corresponding group of linear transformations of the periods.

And we may add as a corollary: Of these linear transformations of the

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* Cf. Klein, Hyperelliptische Sigmafunctionen, Math. Ann. Bd. 27, p. 437.

periods, only those can be congruent to identity modulo 2[*] which correspond to the substitutions
$$z_1 = \pm z_1', \; z_2 = \pm z_2',$$
as we easily infer from the conclusions of §1.

§9.

Conversely, if the system of ϑ-moduli $\tau_{\alpha\beta}$ remains unchanged for a linear transformation s of the periods, not congruent to identity modulo 2, the sextic f remains unchanged for a corresponding linear substitution S of the variables z_1, z_2, different from
$$z_1 = \pm z_1', \; z_2 = \pm z_2'.$$
To prove this we transform the sextic f into the canonical form of Richelot belonging to the system of ϑ-moduli $\tau_{\alpha\beta}$. If, to fix the ideas, we suppose the path of integration K_1 to include the two branching points ("Verzweigungspuncte") α_1 and α_2, K_2 the points α_3 and α_4, K_3 the points α_6 and α_1, K_4 the points α_4 and α_5, Richelot's moduli $\varkappa^2$, λ^2, μ^2 are given by the formulæ
$$\varkappa^2 = \frac{(\alpha_3 - \alpha_4)(\alpha_2 - \alpha_5)}{(\alpha_3 - \alpha_5)(\alpha_2 - \alpha_4)} = \frac{\vartheta_{23}^2 \vartheta_4^2}{\vartheta_{01}^2 \vartheta_5^2},$$
$$\lambda^2 = \frac{(\alpha_3 - \alpha_4)(\alpha_1 - \alpha_5)}{(\alpha_3 - \alpha_5)(\alpha_1 - \alpha_4)} = \frac{\vartheta_{03}^2 \vartheta_{23}^2}{\vartheta_{12}^2 \vartheta_{01}^2},$$
$$\mu^2 = \frac{(\alpha_3 - \alpha_4)(\alpha_6 - \alpha^5)}{(\alpha_3 - \alpha_5)(\alpha_6 - \alpha_4)} = \frac{\vartheta_{03}^2 \vartheta_4^2}{\vartheta_{12}^2 \vartheta_5^2},$$
ϑ_α denoting
$$\vartheta_\alpha = \vartheta(0, 0; \tau_{11}, \tau_{12}, \tau_{22})_\alpha.$$
The sextic f is then transformed into
$$F = \text{Const.}\, y_1 y_2 (y_2 - y_1)(y_2 - \varkappa^2 y_1)(y_2 - \lambda^2 y_1)(y_2 - \mu^2 y_1)$$
by means of the substitution
$$\begin{aligned} y_1 &= (\alpha_3 - \alpha_5)(z_1 - \alpha_4 z_2), \\ y_2 &= (\alpha_3 - \alpha_4)(z_1 - \alpha_5 z_2). \end{aligned} \tag{T}$$
Suppose, now, τ_{11}', τ_{12}', τ_{22}' to be any other system of ϑ-moduli belonging to the same sextic f and derived from τ_{11}, τ_{12}, τ_{22} by means of a linear transformation s of the periods, then the corresponding moduli of Richelot, $\varkappa'^2$, λ'^2, μ'^2, can be derived from $\varkappa^2$, λ^2, μ^2 by applying a certain permutation
$$\sigma = \begin{pmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \alpha_5, & \alpha_6 \\ \alpha_1', & \alpha_2', & \alpha_3', & \alpha_4', & \alpha_5', & \alpha_6' \end{pmatrix}$$

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[*] A linear transformation of the periods is said to be "congruent to identity modulo k," if all the coefficients are divisible by k except those in the principal diagonal, the latter being congruent to 1 (modulo k).

to the roots α_i, and the sextic f passes into the form

$$F' = \text{Const. } y'_1 y'_2 (y'_2 - y'_1)(y'_2 - \varkappa'^2 y'_1)(y'_2 - \lambda'^2 y'_1)(y'_2 - \mu'^2 y'_1)$$

by means of the substitution

$$\begin{aligned} y'_1 &= (\alpha'_3 - \alpha'_5)(z_1 - \alpha'_4 z_2), \\ y'_2 &= (\alpha'_3 - \alpha'_4)(z_1 - \alpha'_5 z_2). \end{aligned} \tag{T'}$$

If, now, we suppose the transformation s to be one of those which leave the $\tau_{\alpha\beta}$ unchanged, so that

$$\tau'_{\alpha\beta} = \tau_{\alpha\beta},$$

we have

$$\vartheta(0,\ 0;\ \tau'_{11},\ \tau'_{12},\ \tau'_{22})_a = \vartheta(0,\ 0;\ \tau_{11},\ \tau_{12},\ \tau_{22})_a,$$

and therefore

$$\varkappa'^2 = \varkappa^2,\ \lambda'^2 = \lambda^2,\ \mu'^2 = \mu^2.$$

Thence we infer that the sextic f is only changed by a constant factor if we apply to the variables z_1, z_2 the combined substitution $T'T^{-1}$. Moreover, if the substitution s is not congruent to identity modulo 2, the permutation σ is different from identity, and therefore the substitution $T'T^{-1}$, considered as non-homogeneous, is different from identity too. Thence the substitution $T'T^{-1}$, considered as homogeneous and reduced to the determinant 1, is different from $z_1 = \pm z'_1$, $z_2 = \pm z'_2$, as was to be proved.

§10.

We now proceed to the application of the method above explained to each of our canonical forms (A).

I. Case: Using non-homogeneous variables

$$z = \frac{z_1}{z_2},$$

we write

$$f(z) = z^6 + \alpha z^4 + \beta z^2 + 1.$$

The roots of $f(z)$ being in pairs equal and of opposite signs, we choose their order so that

$$\alpha_2 = -\alpha_1,\ \alpha_4 = -\alpha_3,\ \alpha_6 = -\alpha_5.$$

The periods may be calculated in the following manner:

Starting from the point α_1, we join the branching points in their natural order by any line L nowhere cutting itself; and we call that side of the line positive which is on the left when we pass along the line in the direction from α_1 through α_2 .... to α_6. We choose as connections* ("Verzweigungsschnitte") of our Riemann's surface T the portions $\alpha_1\alpha_2$, $\alpha_3\alpha_4$, $\alpha_5\alpha_6$ of the line L, and we define as the upper leaf the one for which

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* The expression "connection" has been introduced by Cole.

$$\sqrt{f(0)} = +1.$$

Denoting, then, by $F_{\lambda\mu}$ either of the integrals

$$\int_{\alpha_\lambda}^{\alpha_\mu} \frac{zdz}{\sqrt{f(z)}}, \quad \int_{\alpha_\lambda}^{\alpha_\mu} \frac{dz}{\sqrt{f(z)}},$$

taken along the positive side of the line L in the upper leaf of the surface, ω_i being accordingly ω_{1i} or ω_{2i}, we may write

$$\begin{aligned} \omega_1 &= 2F_{12}, & \omega_2 &= 2F_{34}, \\ \omega_3 &= 2F_{16}, & \omega_4 &= 2F_{45}. \end{aligned}$$

The substitution $$z' = -z$$

leaves $f(z)$ unchanged: $$f(z') = f(z),$$

and the line L is changed into the line L' obtained by turning the whole plane through an angle of 180° around the point $z = 0$; by this operation the branching points are interchanged according to the permutation

$$\sigma = (\alpha_1\alpha_2)(\alpha_3\alpha_4)(\alpha_5\alpha_6),$$

or

$$\alpha_1' = \alpha_2, \ \alpha_2' = \alpha_1, \text{ etc.}$$

We now construct a second Riemann's surface T' with the connections $\alpha_1'\alpha_2'$, $\alpha_3'\alpha_4'$, $\alpha_5'\alpha_6'$ along L', the upper leaf being again defined by

$$\sqrt{f(0)} = +1,$$

and we complete the correspondence of T and T' determined by the substitution $z' = -z$ by requiring that the point $z = 0$, $\sqrt{f(z)} = +1$ of T corresponds to the point $z' = 0$, $\sqrt{f(z')} = +1$ of T'.

Denoting by $\psi(z)$ the value of $\sqrt{f(z)}$ in the point z of the upper leaf of T, and by $\psi'(z)$ the value of $\sqrt{f(z)}$ in the same point z of the upper leaf of T', we easily find the plane of the variable z to be divisible into four annular portions (see Fig. 1), so that $\frac{\psi'(z)}{\psi(z)} = +1$ in the shaded and $= -1$ in the blank portions, the full lines in the figure representing parts of the line L; the dotted, parts of L'.

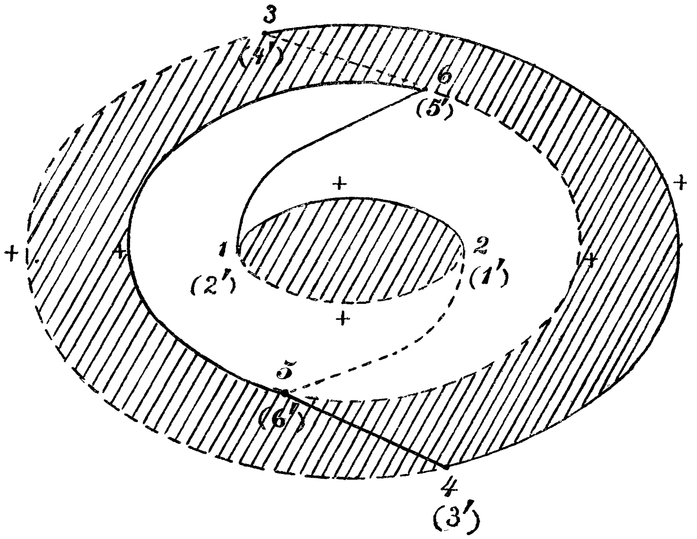

Fig. 1.

We may now easily determine the transformation s. For instance,

$$\omega_1' = 2I'_{1'2'},$$

where $I'_{1'2'}$ extends from $1'$ to $2'$ along the positive side of L' in the upper leaf of T'. We see from the figure that along this path

$$\psi'(z) = -\psi(z);$$

we may therefore calculate $I'_{1'2'}$ also in the surface T along L' *in the lower leaf*, or by deforming gradually the path of integration, along the negative side of L in the lower leaf, that is to say, along the positive side of L in the upper leaf from α_2 to α_1. Therefore

$$\omega_1' = 2I'_{1'2'} = +2I_{21} = -2I_{12} = -\omega_1.$$

In like manner we find

$$\begin{aligned} \omega_2' &= 2I'_{3'4'} = +2I_{34} = \omega_2, \\ \omega_3' &= 2I'_{1'6'} = -2F_{25} = -2F_{16} + 2I_{34} = \omega_2 - \omega_3, \\ \omega_4' &= 2I'_{4'5'} = +2F_{30} = 2F_{45} - 2I_{12} = \omega_4 - \omega_1, \end{aligned}$$

which is, indeed, a "canonical substitution"* of the determinant 1.

The corresponding formulæ for the transformation of the ϑ-moduli are

$$\tau'_{11} = \tau_{11}, \quad \tau'_{12} = 1 - \tau_{12}, \quad \tau'_{22} = \tau_{22},$$

whence we have, putting $\tau'_{\alpha\beta} = \tau_{\alpha\beta}$,

$$\tau_{12} = \frac{1}{2}$$

as the relation between the $\tau_{\alpha\beta}$ which we seek.

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[*] Cf. Clebsch and Gordan, Abel'sche Functionen, p. 300.

The result agrees exactly with the following theorem of Weierstrass* and Picard† concerning the reduction of hyperelliptic to elliptic integrals:

"If any integral of the first species (Gattung) and the first order (Ordnung) be reducible to elliptic integrals by a rational transformation of the k^{th} degree, there can always be found a corresponding system of ϑ-moduli in which

$$\tau_{12} = \frac{1}{k}\,."$$

Applying to the periods the linear transformation

$$\begin{aligned} \omega_1 &= \omega_1 - \omega_4, \\ \bar\omega_2 &= -\omega_4, \\ \bar\omega_3 &= \omega_3, \\ \bar\omega_4 &= \omega_2 - \omega_3, \end{aligned}$$

whence follow

$$\bar\tau_{11} = \frac{\tau_{11}\tau_{22} - \tau_{12}^2}{\tau_{22}}, \quad \bar\tau_{12} = -\frac{\tau_{12} + \tau_{11}\tau_{22} - \tau_{12}^2}{\tau_{22}}, \quad \bar\tau_{22} = \frac{-1 + 2\tau_{12} + \tau_{11}\tau_{22} - \tau_{12}^2}{\tau_{22}};$$

we may give the relation $\tau_{12} = \frac{1}{2}$ also in the form

$$\bar\tau_{11} = \bar\tau_{22},$$

which will be useful afterwards.

§11.

An analogous proceeding being applicable to the other cases, I confine myself to giving for each of them the figure which shows the paths of integration and the value of $\frac{\sqrt{f'(z)}}{\sqrt{f(z)}}$, the linear transformation of the periods and the resulting relation between the $\tau_{\alpha\beta}$.

II. Case: $f(z) = (z^5 + 1)\,z.$

$$\alpha_1 = e^{\frac{-\pi i}{5}}, \quad \alpha_2 = e^{\frac{\pi i}{5}}, \quad \alpha_3 = e^{\frac{3\pi i}{5}}, \quad \alpha_4 = -1, \quad \alpha_5 = e^{\frac{F\pi i}{5}}, \quad \alpha_6 = 0.$$

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* Acta Mathematica, Bd. 4, p. 400.

† Bulletin de la Société mathematique de France, Tome XI.

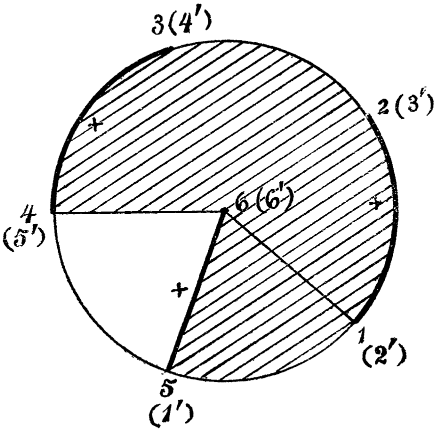

Fig. 2.

$$\varepsilon = e^{\frac{2\pi i}{5}},$$
$$z = \varepsilon z',$$
$$f(z) = \varepsilon f(z') = f'(z'),$$
$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = \begin{cases} + e^{\frac{\pi i}{5}} \text{ in the shaded portion,} \\ - e^{\frac{\pi i}{5}} \text{ in the blank portion.} \end{cases}$$

Thence
$$\omega_1' = \omega_3 - \omega_4,$$
$$\omega_2' = \omega_4,$$
$$\omega_3' = -\omega_1 + \omega_3,$$
$$\omega_4' = -\omega_1 - \omega_2 + \omega_3,$$
and $\tau_{11} = 1 - \varepsilon^4$, $\tau_{12} = -\varepsilon^2 - \varepsilon^4$, $\tau_{22} = \varepsilon$.

III. Case: $f(z) = z(z^4 + az^2 + 1)$.

$$\alpha_5 = 0,\ \alpha_6 = \infty,\ \alpha_3 = -\alpha_1,\ \alpha_4 = -\alpha_2,\ \alpha_1 = \frac{1}{\alpha_2},\ \alpha_3 = \frac{1}{\alpha_4}.$$

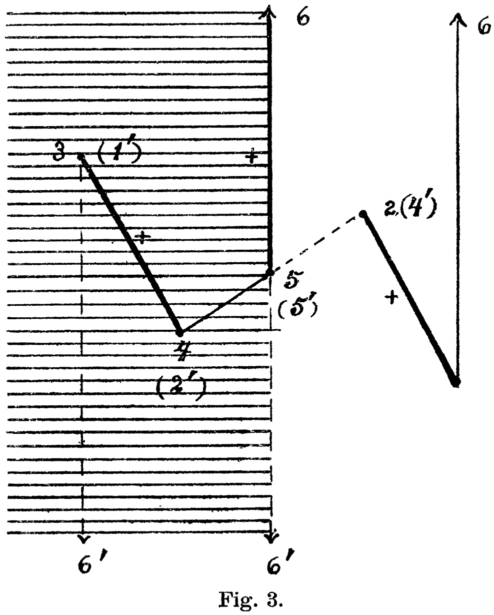

Fig. 3.

$$z = -z',$$
$$f(z) = f'(z') = -f(z'),$$
$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = \pm i,$$

+ in the shaded portion,
— in the blank portion.

Thence
$$\omega_1' = -\omega_2,$$
$$\omega_2' = \omega_1,$$
$$\omega_3' = -\omega_4 + \omega_1,$$
$$\omega_4' = \omega_3 - \omega_2,$$
and $\tau_{12} = \frac{1}{2}$, $\tau_{11} = \tau_{22}$.

IV. Case: $f(z) = z^6 + az^3 + 1$.

$$\left.\begin{matrix} \alpha_4 = \varepsilon\alpha_2, & \alpha_6 = \varepsilon^2\alpha_2 \\ \alpha_3 = \varepsilon\alpha_1, & \alpha_5 = \varepsilon^2\alpha_1 \end{matrix}\right\} \quad \alpha_2 = \frac{1}{\alpha_1},\ \varepsilon = e^{\frac{2\pi i}{3}}.$$

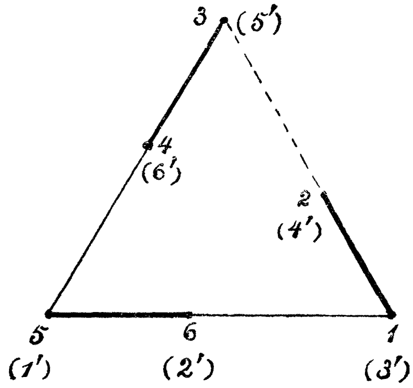

Fig. 4.

$$z = \varepsilon z',$$

$$f(z) = f'(z') = f(z').$$

Here we have in the whole plane

$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = +1.$$

Thence

$$\omega_1' = \omega_2,$$

$$\omega_2' = -\omega_1 - \omega_2,$$

$$\omega_3' = -\omega_3 + \omega_4,$$

$$\omega_4' = -\omega_3,$$

and the resulting relations are

$$\tau_{11} = \tau_{22} = 2\tau_{12},$$

which may also be given in the form

$$\bar{\tau}_{12} = \frac{1}{2}, \; 12\bar{\tau}_{11}\bar{\tau}_{22} + 1 = 0.$$

V. Case: $f(z) = z^6 + 1.$

$$\alpha_1 = e^{\frac{-\pi i}{6}}, \; \alpha_2 = j\alpha_1, \; \alpha_3 = j^2\alpha_1, \; \alpha_4 = j^3\alpha_1, \; \alpha_5 = j^4\alpha_1, \; \alpha_6 = j^5\alpha_1.$$

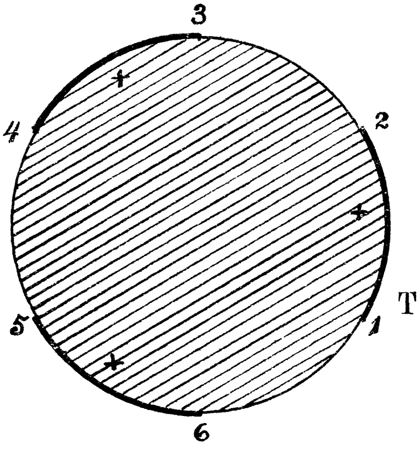

Fig. 5.

$$j = e^{\frac{\pi i}{3}},$$

$$z = jz',$$

$$f(z) = f'(z') = f(z'),$$

$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = \pm 1,$$

+ in the shaded portion,

— in the blank portion.

Thence

$$\omega_1' = \omega_3 - \omega_4,$$

$$\omega_2' = \omega_4,$$

$$\omega_3' = -\omega_1,$$

$$\omega_4' = -\omega_1 - \omega_2.$$

The square of this transformation being identical with the transformation of Case IV, we find

$$\tau_{11} = \tau_{22} = \frac{2}{\sqrt{3}}\,i, \quad \tau_{12} = \frac{1}{\sqrt{3}}\,i;$$

or,

$$\bar{\tau}_{11} = \bar{\tau}_{22} = \frac{i}{2\sqrt{3}}; \quad \bar{\tau}_{12} = \frac{1}{2}.$$

VI. Case: $$f(z) = z(z^4+1).$$

$$\alpha_5 = 0,\ \alpha_6 = \infty,\ \alpha_1 = e^{\frac{-\pi i}{4}},\ \alpha_2 = e^{\frac{\pi i}{4}},\ \alpha_3 = e^{\frac{3\pi i}{4}},\ \alpha_4 = e^{\frac{5\pi i}{4}}.$$

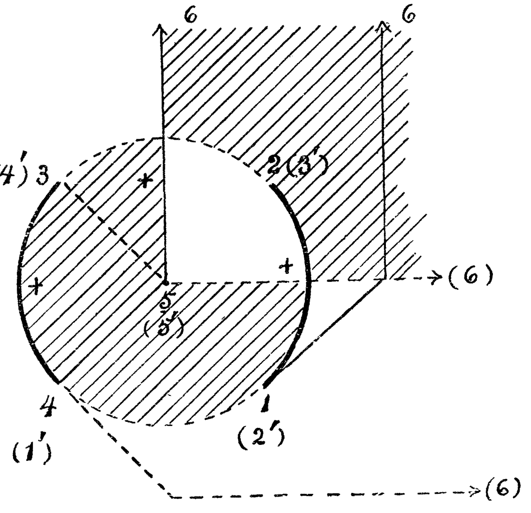

Fig. 6.

$$z = iz',$$

$$f(z) = f'(z') = if(z'),$$

$$\frac{\sqrt{f'(z)}}{\sqrt{f(z)}} = \pm e^{\frac{\pi i}{4}},$$

+ in the shaded portions,
— in the blank portions.

Thence

$$\begin{aligned} \omega_1' &= \omega_3 + \omega_4, \\ \omega_2' &= -\omega_1 + \omega_2 - \omega_3 + \omega_4, \\ \omega_3' &= -\omega_1 - \omega_3, \\ \omega_4' &= \omega_1 - \omega_2 + \omega_3. \end{aligned}$$

The square of this transformation being identical with the transformation of Case III, we find $$\tau_{11} = \tau_{22} = \frac{-1+i\sqrt{2}}{2},\ \tau_{12} = \frac{1}{2}.$$

On comparison of the different cases, the relations between the $\tau_{\alpha\beta}$ for the function $z(z^4+1)$ appear as a special case of the relations for $z(z^4+\alpha z^2+1)$, the latter being themselves a special case of the relations for $z^6+\alpha z^4+\beta z^2+1$, as was to be expected from the fact that the cyclic group $(n=2)$ is contained as a subgroup in the diedron group $(n=2)$, and the latter in the oktaedron group. An analogous remark may be made respecting the three groups, cyclic $n=2$, diedron $n=3$, and diedron $n=6$; and respecting the three groups, cyclic $n=2$, diedron $n=2$, and diedron $n=6$.

A few words may be added concerning the sufficiency of the relations between the $\tau_{\alpha\beta}$. Taking for instance the Case V,

$$\bar{\tau}_{11} = \bar{\tau}_{22} = \frac{i}{2\sqrt{3}},\ \bar{\tau}_{12} = \frac{1}{2},$$

we find first that $\tau_{\alpha\beta}$ remain unchanged for a linear transformation whose period is 6.* Thence we infer as above, §9, that f must remain unchanged for a corresponding substitution of the variables possessing likewise the period 6, and therefore for all the substitutions of the diedron group for $n=6$, q. e. d.

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* Any operation T is said to possess the period μ, if μ is the least possible integer for which $T^{\mu}=1$.

In the case of
$$\bar{\tau}_{12} = \frac{1}{2}\,, \quad 12\bar{\tau}_{11}\bar{\tau}_{22} + 1 = 0\,,$$
we find similarly that f remains unchanged for the substitutions of the diedron group $n = 3$; but, that F may remain unchanged for no other substitution, it is necessary to add a certain inequality, expressing the condition that $\bar{\tau}_{11}$, $\bar{\tau}_{12}$, $\bar{\tau}_{22}$ be not reducible by a linear transformation to the values
$$\bar{\tau}_{11} = \bar{\tau}_{22} = \frac{i}{2\sqrt{3}}\,, \quad \bar{\tau}_{12} = \frac{1}{2}\,,$$
which are characteristic for the diedron group $n = 6$. A similar remark is applicable to the other cases.

We conclude by combining in a table the relations which exist for the sextics with linear substitutions into themselves, on the one hand between the rational invariants, and on the other hand between the $\tau_{\alpha\beta}$.

	Canonical form.	Invariant criteria.	Relations between the $\tau_{\alpha\beta}$.
I.	$z_1^6 + \alpha z_1^4 z_2^2 + \beta z_1^2 z_2^4 + z_2^6$	$R = 0$; $A_{ll}A_{mm} - A_{lm} \gtrless 0$	$\tau_{12} = \frac{1}{2}$
II.	$z_1(z_1^5 + z_2^5)$	$A = 0,\ B = 0,\ C = 0$; $D \gtrless 0$	$\left.\begin{array}{l}\tau_{11} = 1 - \varepsilon^4 \\ \tau_{12} = -\varepsilon^2 - \varepsilon^4 \\ \tau_{22} = \varepsilon\end{array}\right\}\ \varepsilon = e^{\frac{2\pi i}{5}}$
III.	$z_1 z_2 (z_1^4 + \alpha z_1^2 z_2^2 + z_2^4)$	$3AB^2 - 6BC + 4A^2C - 18D = 0$, $4B^3 + 5ABC + 6C^2 - 3AD = 0$; $D \gtrless 0,\ C^2 - \frac{1}{6}B^3 \gtrless 0$	$\tau_{12} = \frac{1}{2}$, $\tau_{11} = \tau_{22}$
IV.	$z_1^6 + \alpha z_1^3 z_2^3 + z_2^6$	$C^2 - \frac{1}{6}B^3 = 0$, $9D - 2B(6C + AB) = 0$; $D \gtrless 0,\ 2AB - 15C \gtrless 0$	$\tau_{12} = \frac{1}{2}$, $12\tau_{11}\tau_{22} + 1 = 0$
V.	$z_1^6 + z_2^6$	$6B - A^2 = 0,\ 6C + AB = 0$, $D = 0$; $A \gtrless 0$	$\tau_{12} = \frac{1}{2}$, $\tau_{11} = \tau_{22} = \frac{i}{2\sqrt{3}}$
VI.	$z_1 z_2 (z_1^4 + z_2^4)$	$B = 0,\ C = 0,\ D = 0$; $A \gtrless 0$	$\tau_{12} = \frac{1}{2}$, $\tau_{11} = \tau_{22} = \frac{-1 + i\sqrt{2}}{2}$

(*B*)

GÖTTINGEN, August, 1887.